CVA for discretely monitored barrier option under stochatic jump model

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Outline

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   - CVA introduction
2. Discretely monitored barrier option
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   - Underlying asset driven process
     - Heston stochastic volatility model
     - Bates stochastic volatility jump model
   - discretely monitored barrier option pricing under Heston and Bates model
   - CVA for discretely monitored barrier option
3. Model risk analysis and numerical results
   - EE and PFE profile for European option and barrier option
   - Impact of parameters
   - Conclusion

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Counterparty Risk and Counterparty Valuation Adjustments(CVA)

Counterparty risk : the exposure to loss due to a specific counterparty failing to meet contractual obligations,i.e.defaultings. If a counterparty defaults,all of their contracts are affected:

- OTC derivatives
- bond issues
- stock issues
- debts,loans

It is worthwhile to think of counterparty risk as the impact of default risk.How does the counterparty exposure and the risk of default impact the value of the security?

- CVA is the expected loss due to counterparty default at any time before portfolio maturity
- CVA is the credit reserve adjustment made to derivatives transactions to account for counterparty risk
- Price of risky security = Risk free price - CVA

How CVA is calculated?

- CVA is the expected value of credit losses over the lifetime of the trade. i.e

$$CVA = (1 - R^{cpty}) \sum_{j=1}^{n} E\left[P(0,t_j) \cdot max(V(t_j),0) \cdot 1_{\{t_{j-\mathbf{1}} < \tau_{cpty} < t_j, \tau_{bank} > t_j\}}\right]$$

R^{cpty}— Counterparty recovery rate, $P(0,t_j)$— Interest rate discount factor

$V(t_j)$— Value of trade at t_j, τ^{cpty},τ^{bank}—Counterparty default time and bank default time

- EE and PFE definition
  - $EE(t_i) = E\{max(V(t_i),0)|\mathscr{F}_0\}$— Positive expected exposure at future time
  - $PFE(t_i) = inf\{x|P(E(t_i) < x|\mathscr{F}_0) > \alpha\}$– PFE measure the worst loss for the risk measure purpose at future time

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Discretely monitored barrier option definition

- Barrier options whose barrier is monitored only at discrete times are called discrete barrier options.
- They are more common than the continuously monitored versions.
- Suppose the monitored times are $t_0 = 0, t_1, t_2, ..., t_n = T$, then down-and-out - barrier option only has value if the observed value stays above the barrier.

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Heston model Introduction

- The Heston stochatic model is given by:

$$
\begin{aligned}
dS(t) &= \mu S(t)dt + \sqrt{V(t)}S(t)dW_1(t) \\
dV(t) &= \kappa(\theta - V(t))dt + \sigma\sqrt{V(t)}dW_2(t) \\
< dW_1(t), dW_2(t) > &= \rho\, dt \\
S(0) &= S_0 \\
V(0) &= V_0
\end{aligned}
$$

- Heston parameters: μ represents the drift of the process, κ is the mean reversion speed for the variance, θ is the mean reversion level, σ is the volatility of the variance and V_0 is the intial level of variance.

Bates model

- The Bates model is given by:

$$
\begin{aligned}
dS(t) &= rS(t)dt + \sqrt{V(t)}S(t)dW_1(t) + (Y-1)S(t)dN(t) \\
dV(t) &= \kappa(\theta - V(t))dt + \sigma\sqrt{V(t)}dW_2(t) \\
< dW_1(t), dW_2(t) > &= \rho dt \\
S(0) &= S_0 \\
V(0) &= V_0 \\
N(0) &= 0
\end{aligned}
$$

The process $N(t)$ is the standard Poisson process which models the numbers of jumps and has intensity λ_J. Y is the jump size distribution which has a log-normal distribution.

$$
Y = \mu_J exp\left(-\frac{1}{2}\sigma_J^2 + \sigma_J Z\right) \;,\; Z \sim N(0,1).
$$

- Bates model is an extension of Heston model. The Heston stochastic volatility model allows for modeling skew and smile shaped implied volatility surfaces. Introducing jumps allows for a significant smile for short dated options. Thus combining the Heston model with a jump model allows for a more pronounced short dated smile shape of implied volatility.

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Discrete monitor barrier option pricing

- Under the Heston and Betas framework, we work in the log-domain, denoted by $(x, v) := (log(S),) \, log(V))$. Suppose that the path values (x_n, v_n) at time t_n are known. Then the option price satisfies the following recursive formula:

$$c(x_n, v_n, t_n) = \begin{cases} e^{-r(t_{n+1}-t_n)} \int_R \int_R c(x_{n+1}, v_{n+1}, t_{n+1}) f_{x,v}(x_{n+1}, v_{n+1}|x_n, v_n|) dx_{n+1} dv_{n+1} & x_n > \xi \\ 0 & x_n \leq \xi \end{cases} \tag{1}$$

- We can write the joint density at t_{n+1} as the following:

$$f_{x,v}(x_{n+1}, v_{n+1}|x_n, v_n) = f_{x|v}(x_{n+1}|v_{n+1}, x_n, v_n) \cdot f_v(v_{n+1}|v_n),$$

where we use $f_{x,v}$ to denote the joint probability density of the log-stock and variance processes at a future time point, given that the information is known at the current time; $f_{x|v}$ denote the probability density of the log-stock process at future time point, given the variance value.

Discrete monitor barrier option pricing

- By the direct link between characteristic function and the Fourier expansion, one obtains a semi-analytic formula which approximate the probability density:

$$f_{x|v}(x_{n+1}|v_{n+1},x_n,v_n) = \frac{2}{b-a}\cdot\sum_{m=0}^{N-1} Re\left\{\phi\left(\frac{m\pi}{b-a};0,v_{m+1},v_m\right)e^{im\pi\frac{x_n-a}{b-a}}\right\}cos\left(m\pi\frac{x_{n+1}-a}{b-a}\right) \tag{2}$$

where

$$[a,b] := \left[c_1 - L\sqrt{|c_2|+\sqrt{|c_4|}}, c_1 + L\sqrt{|c_2|+\sqrt{|c_4|}}\right]$$

and where c_n denotes the $n-th$ cumulant of the log-stock process.

- For the option pricing, double integration is required. By applying the Gauss-quadrature integration rule to the integral, the option pricing formula can be written as:

$$c(x_n,v_n,t_n) := e^{-r\Delta t}\sum_{j=0}^{J-1} w_j \sum_{m=0}^{N-1} V_{m,j}(t_{n+1}) Re\left\{\widetilde{\phi}\left(\frac{m\pi}{b-a};\zeta_j,v_m\right)e^{im\pi\frac{x_n-a}{b-a}}\right\} \tag{3}$$

with

$$V_{m,j} := \frac{2}{b-a}\int_a^b v\left(x_{n+1},\zeta_j,t_{n+1}\right)cos\left(m\pi\frac{x_{n+1}-a}{b-a}\right)dx_{n+1}$$

and w_j are the weights of the quadrature nodes. In order to compute 3, $Re\left\{\widetilde{\phi}\left(\frac{m\pi}{b-a};\zeta_j,v_m\right)e^{im\pi\frac{x_n-a}{b-a}}\right\}$ depends on the characteristic function, the only unknown part is to calculate $V_{m,j}(t_{n+1})$.

Discrete monitor barrier option pricing

- At the maturity date t_M,

$$V_{m,j}(t_M) = \frac{2}{b-a}\int_{[a,l]\cup[u,b]} r_b cos\left(m\pi\frac{y-a}{b-a}\right)dy + \frac{2}{b-a}\int_l^u g(y)cos\left(m\pi\frac{y-a}{b-a}\right)dy \qquad (4)$$

with $g(y)$ defined as follows,

$$g(y) = \alpha K(e^y - 1)^+$$

and $\alpha = 1$ for a call and $\alpha = -1$ for a put option. l and u denote lower and upper barrier levels respectively. Therefore, we can compute the Cosine coefficient analytically at $t = t_M$.

- Given the terminal coefficient $V_{m,j}(t_M)$, a backward recursive calculation can be used to obtain the coefficient at all previous time $t_{M-1}, \cdots, t_2, t_1$.

$$V_{m,j}(t_{M-1}) = \frac{2}{b-a}\int_{[a,l]\cup[u,b]} r_b cos\left(m\pi\frac{y-a}{b-a}\right)dy + \frac{2}{b-a}\int_l^u c(y,\zeta_j,t_{M-1})cos\left(m\pi\frac{y-a}{b-a}\right)dy \qquad (5)$$

$$c(x_n,v_n,t_n) := e^{-r\Delta t} Re\left\{\sum_{m=0}^{N-1}\beta_m(v_n,t_n)e^{im\pi\frac{x_n-a}{b-a}}\right\}$$

where

$$\beta_m(v_n,t_n) := \sum_{m=0}^{N-1} w_j V_{m,j}(t_{n+1})\widetilde{\phi}\left(\frac{m\pi}{b-a};\zeta_j,v_m\right),\ \widetilde{\phi}(w,v_{n+1},v_n) := f_v(v_{n+1}|v_n)\cdot\phi\left(0,e^{v_{n+1}},e^{v_n}\right)$$

Option pricing approximation error

- Truncating the series $\sum_{k=1}^{\infty} A_k(x) V_k \approx \sum_{k=1}^{N-1} A_k(x) V_k$
- Using the characteristic function $\phi(\omega) = \int_R e^{-ix\omega} f(x) dx$ instead of the integral $\int_a^b e^{-ix\omega} f(x) dx$
- To keep the approximation error small we have to choose the truncation range appropriately. We choose for the range of the integration $[a, b]$ such that

$$[a, b] := \left[c_1 - L\sqrt{|c_2| + \sqrt{|c_4|}}, c_1 + L\sqrt{|c_2| + \sqrt{|c_4|}} \right]$$

where c_n is $n-th$ cumulant of $log(S(T)/K$.

Outline

CVA for discretely monitored barrier option

- For the purpose of the CVA calculation, we first use the Monte Carlo simulation to generate market state variables. Next, a grid in S and V is created. on each of the grid, price at each grid are calculated through Cosine expansion method. For any time t which are not in line with simulation time, option price will be obtained by bilinear interpolation on this grid. If the underlying price hit the barrier, the option is exercised at this path and the exposure for later time points are all set to be zero.
- In a summary, the basic procedures are :
  - Generate market scenario for each of the model used;
  - Calculate barrier option values at all future times in the grids.
  - Calculate the expected exposure at each time.
  - Check whether the path hit the barrier. If the underlying price hit the barrier, set the exposure at later time points to be zero.

Calibration of the parameters

In order to fairly compare the model risk between Heston model and Betas models and study the impact of jump, we calibrate a Bates model to the market and Heston model to the benchmark model. Because betas model contains larger number of parameters to calibrate, our benchmark model will be Bates model. Note that for the purpose of study, we consider a stressed market.

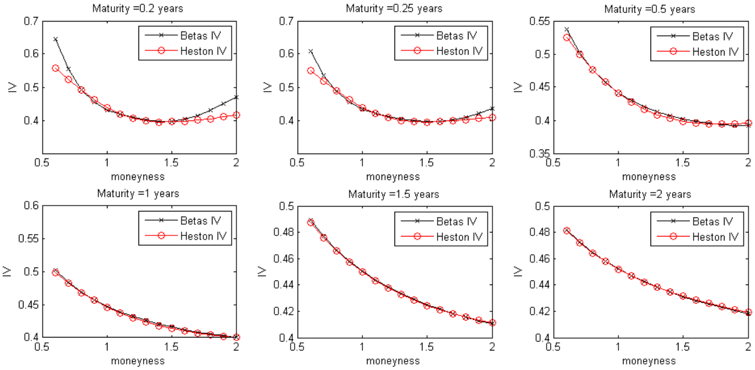
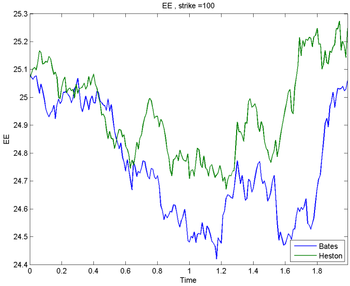


Figure: calibration volatility surface

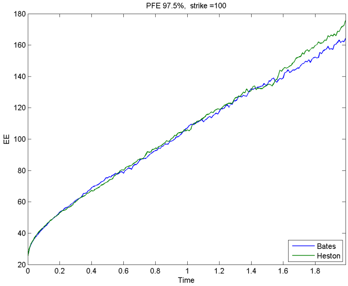
Outline

EE and PFE profile for European option

In order to gain some intuition for the EE and PFE profile, we first show EE and PFE profiles for European option.



(a) EE



(b) PFE

Figure: ATM European option

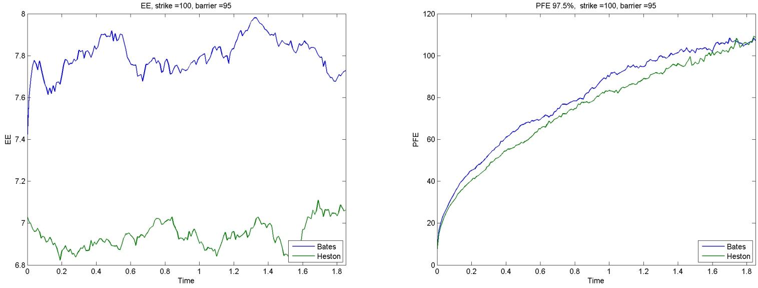
EE and PFE profile for European option

- For European option, EE profile for both Heston and Betas model are relatively close to each other for all OTM, ATM, ITM cases. This behavior is expected since we calibrate Heston models to Betas models.
- EE for both models starts from certain point and oscillates in the lifetime and resumes to the initial level at time to expiry. This part can be explained by the martingale test proposed by Tang Yi Tang.
- PFE profile overlapped with each other at beginning stage , however at the latter lifetime of the option ($t > 1.5$),we see some discrepancy between two models. This difference is partly due to the impact of the model risk. Comparing to Betas model, Heston model may have problems to match the prices of out-of-the-money (OTM) options with short maturities (see, in particular Figure 1,). More often than not, diffusion processes cannot generate the substantial underlying asset movements that are routinely implied by the prices of short-dated OTM options. As option get close to expiration dates, this difference is reflected in the tailed distribution for calculating of PFE.

EE and PFE profile for Barrier option

In this part, we will show the EE and PFE profile for a discretely monitored Barrier option

- Non neglectable difference are observed for EE and PFE for all test cases. The model difference for EE are 10.34%, 11.89% and 13.89% for ITM, ATM and OTM option.
- Similar to European option, EE profile for both models start at certain point and oscillate in the life and resume to the initial level at time to expiry. This part again can be explained by the martingale test theory: when the rebate equals to 0, the discounted cash flow is a martingale



(a) EE (b) PFE

Figure: ATM Barrier option

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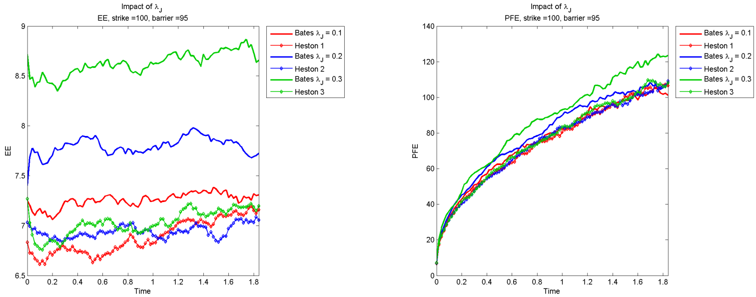
Model difference between Heston model and Betas model

In the following section, we compare the EE for different parameter settings. For all the tests, we examine the model difference between the Betas model and Heston model. We define the model difference for EE profiles between two models as the following:

$$model\ diff = \frac{\sum_{i=1}^{N}\left(abs\left(EE(t_i)^{Betas} - EE(t_i)^{Heston}\right)\right)}{N \cdot \left(\frac{EE(t_0)^{Betas} + EE(t_0)^{Heston}}{2}\right)} \tag{6}$$

where $EE(t_i)^{Betas}$ and $EE(t_i)^{Heston}$ are the EE values obtained by the cosine expansion method at the time t_i, $EE(t_0)^{Betas}$ and $EE(t_0)^{Heston}$ are the current value $(t = 0)$ of the option for Betas and Heston model.

Impact of λ_J



(a) EE

(b) PFE

Figure: Strike = 100

Impact of λ_J

Impact of λ_J	$\lambda_J = 0.1$	$\lambda_J = 0.2$	$\lambda_J = 0.3$
strike = 100	5.33%	11.89%	19.98%
strike = 120	6.6%	13.68%	22.91%

Table: Impact of λ_J

Plots from the Figures suggest that: λ_J have a significant impact on EE and PFE. In general , increasing the jump intensity will increase the model differences. As we know, jump intensityλ_J represent average number of jumps per unit of time. With the increase of the jump frequency, the Heston model fail to catch some jump feather of the fitting and cause large model difference.

Impact of σ_J and Impact of μ_J

Impact of σ_J	$\sigma_J = 0.2$	$\sigma_J = 0.4$	$\sigma_J = 0.55$
strike = 100	12.83%	11.89%	12.82%
strike = 120	13.68%	13.68%	15.99%

Table: Impact of σ_J

Impact of μ_J	$\mu_J = -0.5$	$\mu_J = -0.2$	$\mu_J = 0$
strike = 100	11.89%	4.93%	1.56%
strike = 120	13.68%	6.65%	3.77%

Table: Impact of μ_J

Impact of σ and Impact of ρ

Impact of σ	$\sigma = 0.25$	$\sigma = 0.4$
strike = 100	11.89%	12.16%
strike = 120	13.86%	13.24%

Table: Impact of σ

Impact of ρ	$\rho = -0.5$	$\rho = 0$	$\rho = 0.5$
strike = 100	11.89%	9.16%	12.97%
strike = 120	13.68%	12.42%	16.61%

Table: Impact of ρ

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Conclusion

Comparing to Betas model , Heston model could provide a good fit for long term option, however Heston model cannot generate the substantial underlying asset movements that are routinely implied by the prices of short-dated options. Since Betas model allow jumps to the underlying process,it allows for a significant smile for short dated options.

- Our results show that the impact of jump intensity, volatility of jump size and volatility of volatility parameters could result in unnegelectable model difference between Betas model and Heston model.
- The jump mean and correction parameters that control the skew of the distribution also contribute to noticeable model differences.
- All the numerical analysis show that the stochastic volatility and jump which are widely ignored by practitioners play an important role in evaluating CVA and expected exposures, particularly for stressed market conditions.